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Level-0 structure of level-1 $U_q(\widehat{sl}_2)$ -modules and Macdonald polynomials

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Abstract. The level-1 integrable highest weight modules of $U_q(\widehat{sl}_2)$ admit a level-0 action of the same algebra. This action is defined using the affine Hecke algebra and the basis of the level-1 module generated by components of vertex operators. Each level-1 module is a direct sum of finite-dimensional irreducible level-0 modules, whose highest weight vector is expressed in terms of Macdonald polynomials. This decomposition leads to the fermionic character formula for the level-1 modules.

1. Introduction

The present article is concerned with a curious finding in representation theory, noted recently in connection with conformal field theory (CFT) and solvable lattice models. In the context of quantum affine algebras, the statement is the following. Let $V(\Lambda_i)$ ($i = 0, 1$) be a level-1 integrable highest weight module of $U_q(\widehat{sl}_2)$. Then it admits a level-0 action of the same algebra $U_q(\widehat{sl}_2)$. This second action leaves invariant each homogeneous component $V(\Lambda_i)_{-n}$, and the whole module becomes a direct sum of finite-dimensional irreducible constituents, which can be described explicitly.

The origin of this observation goes back to the study of spin chains with long-range interactions (the Haldane–Shastry model) [11]. This model has a remarkable property that the Yangian $Y(sl_n)$ acts as an exact symmetry even for finite chains. By considering the continuum limit, it was suggested in [11], and subsequently confirmed in [2, 3, 5, 19], that there is an action of the Yangian on level-1 integrable modules of the affine Lie algebra \widehat{sl}_n . This action is related to the fermionic character formula for the level-1 module conjectured in [16]. Similar results are expected to hold for higher level representations as well [4]. The fermionic expressions conjectured there for the characters of the level- k modules, subsequently proved in [1, 17], strongly support the validity of this picture.

What we discuss in this paper is a q -deformation of these structures in the simplest case of $U_q(\widehat{sl}_2)$ with level-1. Namely we consider level-1 modules of $U_q(\widehat{sl}_2)$, and define a level-0 action on them. The quantum affine algebra with level 0 plays the role of the Yangian in CFT. It is not yet clear whether such a q -deformation is related to some physical models like the Haldane–Shastry chain. Apart from the technical complexity, the essence of the construction does not differ very much from the conformal case. In this respect we are not claiming any sort of methodical novelty. Our aim here is to supply the mathematical details and give a coherent account of this as yet mysterious phenomenon.

Here is the plan of the paper. In section 2 we introduce a level-0 action of $U_q(\widehat{sl}_2)$ on the sum of level-1 integrable highest weight modules $\mathcal{H} = V(\Lambda_0) \oplus V(\Lambda_1)$. This part is a review of our previous paper [12]. The level-0 action is defined with the help of the affine Hecke algebras which appear in [2], and the action is given on certain generating series of the vectors in \mathcal{H} . To ensure the construction is well-defined, we introduce appropriate completions of the spaces, and sort out all the linear relations among the generating series mentioned above. In the conformal case, the Yangian generators have explicit realizations in terms of currents. Lacking such formulae in the deformed case, our construction is rather indirect. In section 3 we define a family of highest weight vectors $\omega_{\lambda,N} \in \mathcal{H}$ with the aid of Macdonald polynomials, following the ideas in [3]. These vectors are indexed by $N \in \mathbb{Z}_{\geq 0}$ and a partition λ of length at most N . We remark that the vectors $\omega_{\lambda,N}$ can be given an explicit formula in terms of bosons and Macdonald symmetric functions (see (3.9)). This point will not be used in the rest of the paper. We show in section 4 that $\omega_{\lambda,N}$'s generate inside \mathcal{H} irreducible modules of the form (see (4.5)–(4.7))

$$W_{\lambda,N} = W_{n_1}(a_1) \otimes \cdots \otimes W_{n_m}(a_m).$$

Here $W_n(a)$ is the $(n + 1)$ -dimensional evaluation module (see appendix A for conventions). Moreover \mathcal{H} is a direct sum of them:

$$V(\Lambda_i) = \bigoplus_{\substack{N \equiv i \pmod{2} \\ l(\lambda) \leq N}} W_{\lambda,N}.$$

This decomposition corresponds to the known fermionic sum formula for the characters

$$\text{ch } V(\Lambda_i) = \text{tr}_{V(\Lambda_i)}(q^d) = \sum_{\substack{N \equiv i \pmod{2} \\ l(\lambda) \leq N}} q^{(N^2-i)/4+|\lambda|} \prod_{j=1}^m (n_j + 1)$$

which we use in the course of the proof of these statements.

Throughout this paper we fix a complex number q such that $0 < |q| < 1$. In section 4 we assume that q is transcendental over \mathbb{Q} .

2. Level-0 action

In this section we define the level-0 action on the level-1 integrable modules of $U_q(\widehat{sl}_2)$. The proofs of the statements in this section are given in [12].

2.1. Spinon basis

This subsection contains some preliminary material. Let us fix the notation following [13]. The quantum affine algebra $U = U_q(\widehat{sl}_2)$ is generated by $e_i, f_i, t_i = q^{h_i}$ ($i = 0, 1$) and q^d . When we refer to the subalgebra $U_q(sl_2)$, we mean the one generated by e_1, f_1 and t_1 . By U' we mean the subalgebra of U generated by e_i, f_i, t_i ($i = 0, 1$). We use the coproduct Δ given by

$$\begin{aligned} \Delta(e_i) &= e_i \otimes 1 + t_i \otimes e_i & \Delta(f_i) &= f_i \otimes t_i^{-1} + 1 \otimes f_i \\ \Delta(q^h) &= q^h \otimes q^h & (h &= h_i, d). \end{aligned}$$

Let $V = Cv_+ \oplus Cv_-$ be the irreducible two-dimensional module of $U_q(sl_2)$. Let further

$$V_{\text{aff}} = \text{span}_{\mathbb{C}}\{v_{\varepsilon,n} \mid \varepsilon = \pm, n \in \mathbb{Z}\}$$

be the level-0 U -module defined as follows.

$$\begin{aligned} e_0 v_{+,n} &= v_{-,n+1} & e_0 v_{-,n} &= 0 & f_0 v_{+,n} &= 0 & f_0 v_{-,n} &= v_{+,n-1} \\ e_1 v_{+,n} &= 0 & e_1 v_{-,n} &= v_{+,n} & f_1 v_{+,n} &= v_{-,n} & f_1 v_{-,n} &= 0 \\ t_0^{-1} v_{\pm,n} &= t_1 v_{\pm,n} = q^{\pm 1} v_{\pm,n} & q^d v_{\pm,n} &= q^n v_{\pm,n}. \end{aligned}$$

We shall use the generating series

$$v_\varepsilon(z) = \sum_n v_{\varepsilon,n} z^{-n}.$$

Let $V(\Delta_i) = U|i\rangle$ ($i = 0, 1$) be the integrable level-1 U -module with highest weight vector $|i\rangle$. The index i is to be read modulo 2, i.e. $\Delta_{i+2} = \Delta_i$. Set $\mathcal{H} = V(\Delta_0) \oplus V(\Delta_1)$. This is a \mathbb{Z} -graded vector space

$$\mathcal{H} = \bigoplus_{r \leq 0} \mathcal{H}_r \quad \mathcal{H}_r = \{u \in \mathcal{H} | q^d u = q^r u\}.$$

In particular, $\mathcal{H}_0 = \mathbb{C}|0\rangle \oplus \mathbb{C}|1\rangle$. We regard $\text{End}_{\mathbb{C}} \mathcal{H}$ as a level-0 U -module by the opposite adjoint action

$$\text{ad}^{\text{op}} x \cdot f = \sum x_{(2)} f a^{-1}(x_{(1)})$$

where $f \in \text{End}_{\mathbb{C}} \mathcal{H}$, $x \in U$, $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$ and a denotes the antipode.

The level-0 action will be defined with the aid of the type-I vertex operator

$$\tilde{\Phi}_\varepsilon^*(z) = \sum_{n \in \mathbb{Z}} \tilde{\Phi}_{\varepsilon,n}^* z^{-n}.$$

By definition, it is the generating series of operators $\tilde{\Phi}_{\varepsilon,n}^* \in \text{End}_{\mathbb{C}} \mathcal{H}$ with the following properties: (i) the map $u \otimes v_{\varepsilon,n} \mapsto \tilde{\Phi}_{\varepsilon,n}^* u$ from $\mathcal{H} \otimes V_{\text{aff}}$ to \mathcal{H} is an intertwiner of U -modules, (ii) $\tilde{\Phi}_{\varepsilon,n}^* \mathcal{H}_r \subset \mathcal{H}_{r+n}$, (iii) $\tilde{\Phi}_{+,0}^*|0\rangle = |1\rangle$, $\tilde{\Phi}_{-,0}^*|1\rangle = |0\rangle$.

Consider the vector space

$$\mathcal{V} = \bigoplus_{N \geq 0} \mathcal{V}_N \quad \mathcal{V}_N = V_{\text{aff}}^{\otimes N}$$

with the U -action given by the opposite coproduct $\Delta^{\text{op}}(x) = \sum x_{(2)} \otimes x_{(1)}$.

Proposition 2.1. The map

$$\rho_0 : \mathcal{V} \longrightarrow \mathcal{H} \tag{2.1}$$

$$v_{\varepsilon_1, n_1} \otimes \cdots \otimes v_{\varepsilon_N, n_N} \mapsto \tilde{\Phi}_{\varepsilon_1, n_1}^* \cdots \tilde{\Phi}_{\varepsilon_N, n_N}^* |0\rangle \tag{2.2}$$

is surjective and $U_q(\widehat{sl}_2)$ -linear.

The vectors of the form (2.2) are not linearly independent. As shown in [10], it is possible to choose a suitable subset to construct a basis. A similar basis appeared also in CFT under the name of the spinon basis, which was used for constructing the Yangian action on conformal blocks [3, 4, 19]. We follow the same strategy in the q -deformed case in order to define the level-0 action. For this purpose we need to determine all the linear relations among (2.2) [10]. This will be discussed in the next subsection.

2.2. Linear relations

The map ρ_0 factors through the map

$$\rho : \mathcal{V} \longrightarrow \text{End}_{\mathbb{C}} \mathcal{H} \tag{2.3}$$

$$v_{\varepsilon_1, n_1} \otimes \cdots \otimes v_{\varepsilon_N, n_N} \mapsto \tilde{\Phi}_{\varepsilon_1, n_1}^* \cdots \tilde{\Phi}_{\varepsilon_N, n_N}^* \tag{2.4}$$

Among the linear relations for the vectors (2.2), the major ones derive from the commutation relations of vertex operators. The latter involve infinite sums, and to justify them we need to complete the space \mathcal{V} .

Set

$$\mathcal{V}_N^{(r)} = \text{span}_{\mathbb{C}}\{v_{\varepsilon_1, m_1} \otimes \cdots \otimes v_{\varepsilon_N, m_N} \mid m_1 + \cdots + m_N = r\}.$$

Define a filtration of $\mathcal{V}_N^{(r)}$ by

$$\mathcal{V}_N^{(r)} \supset \cdots \supset \mathcal{V}_N^{(r)}[l] \supset \mathcal{V}_N^{(r)}[l+1] \supset \cdots$$

$$\mathcal{V}_N^{(r)}[l] = \text{span}_{\mathbb{C}}\{v_{\varepsilon_1, m_1} \otimes \cdots \otimes v_{\varepsilon_N, m_N} \in \mathcal{V}_N^{(r)} \mid \max(m_1 + m_2 + \cdots + m_N, \dots, m_{N-1} + m_N, m_N) \geq l\}.$$

We complete the space $\mathcal{V}_N^{(r)}$ with respect to this filtration

$$\hat{\mathcal{V}}_N^{(r)} = \varinjlim_l \mathcal{V}_N^{(r)} / \mathcal{V}_N^{(r)}[l]$$

and set $\hat{\mathcal{V}}_N = \bigoplus_{r \in \mathbb{Z}} \hat{\mathcal{V}}_N^{(r)}$, $\hat{\mathcal{V}}' = \bigoplus_{N \geq 0} \hat{\mathcal{V}}_N'$. Since $\mathcal{H}_t = 0$ for $t > 0$, we have

$$\rho \left(\mathcal{V}_N^{(r)}[l] \right) \mathcal{H}_s = 0 \quad \text{if } l + s > 0.$$

Therefore, the map ρ and ρ_0 extend to the completion:

$$\hat{\rho}' : \hat{\mathcal{V}}' \longrightarrow \text{End}_{\mathbb{C}} \mathcal{H} \quad \hat{\rho}'_0 : \hat{\mathcal{V}}' \longrightarrow \mathcal{H}.$$

To state the commutation relations, we prepare the R -matrix $\tilde{R}(z) \in \text{End}_{\mathbb{C}} V \otimes V$:

$$\begin{aligned} \tilde{R}(z)v_{\varepsilon} \otimes v_{\varepsilon} &= \frac{z - q^2}{1 - q^2z} v_{\varepsilon} \otimes v_{\varepsilon} \\ \tilde{R}(z)v_{+} \otimes v_{-} &= \frac{(1 - q^2)z}{1 - q^2z} v_{+} \otimes v_{-} + \frac{q(z - 1)}{1 - q^2z} v_{-} \otimes v_{+} \\ \tilde{R}(z)v_{-} \otimes v_{+} &= \frac{q(z - 1)}{1 - q^2z} v_{+} \otimes v_{-} + \frac{1 - q^2}{1 - q^2z} v_{-} \otimes v_{+}. \end{aligned} \tag{2.5}$$

These formulae are regarded as a power series in z . Let $\pi_j : \text{End}_{\mathbb{C}} V \rightarrow \text{End}_{\mathbb{C}} V^{\otimes N}$ be the natural injection to the j th component of $\text{End}_{\mathbb{C}} V^{\otimes N} \simeq \text{End}_{\mathbb{C}} V \otimes \cdots \otimes \text{End}_{\mathbb{C}} V$. We set $\tilde{R}_{j,k}(z) = (\pi_j \otimes \pi_k)(\tilde{R}(z))$.

Consider the generating series in $\hat{\mathcal{V}}_N'$,

$$\begin{aligned} F_{\varepsilon_1, \dots, \varepsilon_N}(z_1, \dots, z_N) &= \frac{\prod_{j=1}^N z_j^{(N-j-p_j+p_N)/2}}{\prod_{j < k} \eta(z_k/z_j)} v_{\varepsilon_1}(z_1) \otimes \cdots \otimes v_{\varepsilon_N}(z_N) \\ &= \sum_{m_1, \dots, m_N \in \mathbb{Z}} F_{\varepsilon_1, \dots, \varepsilon_N, m_1, \dots, m_N} z_1^{-m_1} \cdots z_N^{-m_N} \end{aligned} \tag{2.6}$$

where

$$\eta(z) = \frac{(q^6z; q^4)_{\infty}}{(q^4z; q^4)_{\infty}} \quad (z; p)_{\infty} = \prod_{n=0}^{\infty} (1 - p^n z).$$

In (2.6) we have set $p_j = 0$ (if $j \equiv N \pmod{2}$), $= 1$ (otherwise).

Proposition 2.2. The Laurent coefficients of the following belong to $\text{Ker } \hat{\rho}'$.

$$F_{\varepsilon_1, \dots, \varepsilon_j, \varepsilon_{j+1}, \dots, \varepsilon_N}(z_1, \dots, z_{j+1}, z_j, \dots, z_N) - \tilde{R}_{j, j+1}(z_{j+1}/z_j) F_{\varepsilon_1, \dots, \varepsilon_j, \varepsilon_{j+1}, \dots, \varepsilon_N}(z_1, \dots, z_j, z_{j+1}, \dots, z_N). \tag{2.7}$$

Proposition 2.2 implies that the vectors $\hat{\rho}'_0(F_{\varepsilon_1, \dots, \varepsilon_N, m_1, \dots, m_N})$ with $m_1 \leq \dots \leq m_N$ span the space \mathcal{H} .

Let us introduce a second filtration in $\hat{\mathcal{V}}_N^{(r)}$.

$$\begin{aligned} \hat{\mathcal{V}}_N^{(r)} \supset \dots \supset \hat{\mathcal{V}}_N^{(r)}[[m]] \supset \hat{\mathcal{V}}_N^{(r)}[[m+1]] \supset \dots \\ \hat{\mathcal{V}}_N^{(r)}[[m]] = \text{cl. span}_{\mathbb{C}}\{F_{\varepsilon_1, \dots, \varepsilon_N, m_1, \dots, m_N} \in \hat{\mathcal{V}}_N^{(r)} \mid \max(m_1, \dots, m_N) \geq m\} \end{aligned}$$

where cl stands for the closure in $\hat{\mathcal{V}}'$. We denote by $\hat{\mathcal{V}}_N^{(r)}$ the completion of $\hat{\mathcal{V}}_N^{(r)}$ with respect to this filtration

$$\hat{\mathcal{V}}_N^{(r)} = \varprojlim_m \hat{\mathcal{V}}_N^{(r)} / \hat{\mathcal{V}}_N^{(r)}[[m]]$$

and set $\hat{\mathcal{V}}_N = \bigoplus_{r \in \mathbb{Z}} \hat{\mathcal{V}}_N^{(r)}$, $\hat{\mathcal{V}} = \bigoplus_{N \geq 0} \hat{\mathcal{V}}_N$.

From proposition 2.2 it follows that

$$\hat{\rho}'(\hat{\mathcal{V}}_N^{(r)}[[m]]) \mathcal{H}_s = 0$$

if $m + s > 0$. Therefore the maps $\hat{\rho}'$ and $\hat{\rho}'_0$ extend further to $\hat{\mathcal{V}}$:

$$\hat{\rho} : \hat{\mathcal{V}} \longrightarrow \text{End}_{\mathbb{C}} \mathcal{H} \quad \hat{\rho}_0 : \hat{\mathcal{V}} \longrightarrow \mathcal{H}.$$

The vectors $\hat{\rho}'_0(F_{\varepsilon_1, \dots, \varepsilon_N, m_1, \dots, m_N})$ are subject further to two kinds of relations: the highest weight condition (2.8) and the fusion relation (2.9).

Proposition 2.3. The kernel of $\hat{\rho}_0$ contains

$$F_{\varepsilon_1, \dots, \varepsilon_N, m_1, \dots, m_N} \quad \text{with } m_j > 0 \text{ for some } j. \tag{2.8}$$

Proposition 2.4. The Laurent coefficients of the following belong to $\text{Ker } \hat{\rho}$.

$$\begin{aligned} F_{\varepsilon_1, \dots, \varepsilon_N}(z_1, \dots, z_N) \Big|_{z_{j+1}=q^{-2}z_j} - (-q)^{N-j+(\varepsilon_j-1)/2} \delta_{\varepsilon_j+\varepsilon_{j+1}, 0} \\ \times \prod_{i=1}^{j-1} (z_i - q^2 z_j) \prod_{i=j+2}^N (q^{-2} z_j - q^2 z_i) \\ \times F_{\varepsilon_1, \dots, \varepsilon_{j-1}, \varepsilon_{j+2}, \dots, \varepsilon_N}(z_1, \dots, z_{j-1}, z_{j+2}, \dots, z_N). \end{aligned} \tag{2.9}$$

Define $\tilde{\mathcal{N}}$ (respectively \mathcal{N}) to be the closure of the span of elements (2.7) and (2.8) (respectively (2.7), (2.8) and (2.9)) in $\hat{\mathcal{V}}$.

Proposition 2.5. The following map induced from $\hat{\rho}_0$ is a $U_q(sl_2)$ -linear isomorphism

$$\hat{\mathcal{V}}/\mathcal{N} \simeq \mathcal{H}.$$

2.3. Affine Hecke algebra

To distinguish different actions of U' on the same module, we shall sometimes write $U'_{k=l}$ for $l \in \mathbb{C}$ to mean the quotient of U' modulo $q^c = q^l$. We will construct an action of $U'_{k=0}$ on \hat{V} in such a way that the subspace \tilde{N} is invariant under the action. The main tools are the affine Hecke algebras and the L operators acting on \hat{V} .

Define $S \in \text{End}_{\mathbb{C}} V \otimes V$ by

$$\begin{aligned} Sv_{\varepsilon} \otimes v_{\varepsilon} &= -q^{-1}v_{\varepsilon} \otimes v_{\varepsilon} \\ Sv_{+} \otimes v_{-} &= (q - q^{-1})v_{+} \otimes v_{-} - v_{-} \otimes v_{+} \\ Sv_{-} \otimes v_{+} &= -v_{+} \otimes v_{-}. \end{aligned}$$

The operators $S_{j,j+1} = (\pi_j \otimes \pi_{j+1})(S) \in \text{End}_{\mathbb{C}} V^{\otimes N}$ ($j = 1, \dots, N - 1$) satisfy the Hecke algebra relations

$$S_{j,j+1} - S_{j,j+1}^{-1} = q - q^{-1} \tag{2.10}$$

$$S_{j,j+1}S_{k,k+1} = S_{k,k+1}S_{j,j+1} \quad (|j - k| > 1) \tag{2.11}$$

$$S_{j,j+1}S_{j+1,j+2}S_{j,j+1} = S_{j+1,j+2}S_{j,j+1}S_{j+1,j+2}. \tag{2.12}$$

The R -matrix $\tilde{R}(z)$ (2.5) can be written as

$$\tilde{R}(z) = \frac{Sz - S^{-1}}{qz - q^{-1}}. \tag{2.13}$$

Define

$$G_{j,k}^{\pm 1} = \frac{q^{-1}z_j - qz_k}{z_j - z_k}(K_{j,k} - 1) + q^{\pm 1}. \tag{2.14}$$

Here $K_{j,k}$ signifies the exchange of variables z_j and z_k . The $G_{j,j+1}$ ($j = 1, \dots, N - 1$) also satisfy the relations (2.10)–(2.12).

Proposition 2.6. In terms of $S_{j,j+1}$ and $G_{j,j+1}$, the relation (2.7) is written as

$$(S_{j,j+1} - G_{j,j+1})F_{\varepsilon_1, \dots, \varepsilon_N}(z_1, \dots, z_N) = 0. \tag{2.15}$$

The action of the Hecke algebra given via $G_{j,j+1}$ can be extended to that of the affine Hecke algebra. Set

$$\begin{aligned} Y_j &= (G_{j,j+1}^{-1}K_{j,j+1}) \cdots (G_{j,N}^{-1}K_{j,N})p^{\theta_j}(K_{1,j}G_{1,j}) \cdots (K_{j-1,j}G_{j-1,j}) \\ &= G_{j,j+1}^{-1} \cdots G_{N-1,N}^{-1}ZG_{1,2} \cdots G_{j-1,j} \end{aligned} \tag{2.16}$$

where $Z = K_{1,2}K_{1,3} \cdots K_{1,N}p^{\theta_1}$ and p^{θ_j} denotes the scale operator

$$p^{\theta_j}f(z_1, \dots, z_N) = f(z_1, \dots, pz_j, \dots, z_N).$$

The operators $G_{j,j+1}$ ($j = 1, \dots, N - 1$) and Y_j ($j = 1, \dots, N$) satisfy the relations for the affine Hecke algebra \hat{H}_N . Namely we have, in addition to (2.10)–(2.12),

$$\begin{aligned} Y_j Y_k &= Y_k Y_j \\ G_{j,j+1} Y_j G_{j,j+1} &= Y_{j+1} \\ [G_{j,j+1}, Y_k] &= 0 \quad (j, j + 1 \neq k). \end{aligned}$$

We note that the symmetric polynomials in Y_j ($j = 1, \dots, N$) belong to the centre of \hat{H}_N .

For an operator $X \in \text{End}_{\mathbb{C}} \mathbb{C}[z_1, z_1^{-1}, \dots, z_N, z_N^{-1}]$ we define $\hat{X} \in \text{End}_{\mathbb{C}} \hat{V}_N$ by

$$\sum \left(\hat{X} F_{\varepsilon_1, \dots, \varepsilon_N; m_1, \dots, m_N} \right) z_1^{-m_1} \cdots z_N^{-m_N} = \sum F_{\varepsilon_1, \dots, \varepsilon_N; m_1, \dots, m_N} X \left(z_1^{-m_1} \cdots z_N^{-m_N} \right).$$

2.4. L -operators

In order to define $U'_{k=0}$ -action on \widehat{V} we use the L -operator formalism. Some of the basic facts on L -operators are given in appendix A.

Noting that the operators \widehat{Y}_j ($j = 1, \dots, N$) commute with each other, we set

$$\begin{aligned} \overline{L}_0(x) &= \overline{L}_{01}(x; q^{N-1}\widehat{Y}_1^{-1}) \cdots \overline{L}_{0N}(x; q^{N-1}\widehat{Y}_N^{-1}) \\ \overline{L}_{0j}(x; a) &= \frac{xS_{0j}^{-1} - qaS_{0j}}{qa - x} P_{0j}. \end{aligned} \tag{2.17}$$

Define $l_{jk}^{\pm}[n] \in \text{End}_{\mathbb{C}} \widehat{V}$ as in (A.1) by expanding $\overline{L}_0(x)$ in $x^{\pm 1}$. The Yang-Baxter equation for $\overline{R}(x)$ (A.2) implies

$$\overline{R}_{0\mathcal{O}'}(x_0/x_{\mathcal{O}'})\overline{L}_0(x_0)\overline{L}_{\mathcal{O}'}(x_{\mathcal{O}'}) = \overline{L}_{\mathcal{O}'}(x_{\mathcal{O}'})\overline{L}_0(x_0)\overline{R}_{0\mathcal{O}'}(x_0/x_{\mathcal{O}'}).$$

Therefore, we get a representation of \widehat{U}' . We denote it by

$$\pi^{(N)} : \widehat{U}' \rightarrow \text{End}_{\mathbb{C}} \widehat{V}.$$

On $U_q(\widehat{sl}_2)$, $\pi^{(N)}$ is equal to the representation induced by the opposite coproduct. We have (cf [7])

$$\pi^{(N)}(e_0) = \sum_{j=1}^N q^{N-1}\widehat{Y}_j^{-1}\pi_j(f_1)\pi_{j+1}(t_1^{-1}) \cdots \pi_N(t_1^{-1}) \tag{2.18}$$

$$\pi^{(N)}(f_0) = \sum_{j=1}^N q^{1-N}\widehat{Y}_j\pi_1(t_1) \cdots \pi_{j-1}(t_1)\pi_j(e_1) \tag{2.19}$$

$$\pi^{(N)}(t_0) = \pi_1(t_1^{-1}) \cdots \pi_N(t_1^{-1}). \tag{2.20}$$

Here we consider

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad t_1 = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

in $\text{End}_{\mathbb{C}} V_{\text{aff}}$ and embed them by $\pi_j : \text{End}_{\mathbb{C}} V_{\text{aff}} \rightarrow \text{End}_{\mathbb{C}} \widehat{V}$.

Because $Y_j^{\pm 1}$ maps polynomials in z to polynomials, the linear span of (2.8) is invariant by $\pi^{(N)}$. Because we have

$$\begin{aligned} \overline{L}_{0j}(x; q^{N-1}\widehat{Y}_j^{-1})\overline{L}_{0j+1}(x; q^{N-1}\widehat{Y}_{j+1}^{-1}) &\left(S_{j,j+1} - \widehat{G}_{j,j+1}\right) \\ &= \left(S_{j,j+1} - \widehat{G}_{j,j+1}\right)\overline{L}_{0j}(x; q^{N-1}\widehat{Y}_{j+1}^{-1})\overline{L}_{0j+1}(x; q^{N-1}\widehat{Y}_j^{-1}) \end{aligned}$$

the linear span of (2.7) is also invariant. Therefore we have

Proposition 2.7. The $U'_{k=0}$ -action $\pi^{(N)}$ on \widehat{V} induces a $U'_{k=0}$ -action on $\widehat{V}/\widetilde{N}$.

In [12] we have further shown that

Proposition 2.8. The $U'_{k=0}$ -action on $\widehat{V}/\widetilde{N}$ induces a $U'_{k=0}$ -action on \widehat{V}/N .

3. Highest weight vectors

3.1. Vectors $\tilde{\omega}_{\lambda,N}$

In this subsection we fix $N \geq 1$ and work with the space \hat{V}/\hat{N} . Following the ideas of [3], we shall introduce a family of highest weight vectors $\tilde{\omega}_{\lambda,N}$ with respect to the level-0 action, and find their Drinfeld polynomials. In what follows we shall write z for (z_1, \dots, z_N) .

Set $\gamma(z) = \prod_{j < k} (z_j - q^2 z_k)$, and define a generating series in \hat{V}/\hat{N} ,

$$\Omega(z) = \gamma(z)^{-1} F_{+\dots+}(z) \tag{3.1}$$

$$= \frac{\prod_{j=1}^N z_j^{(j-N-p_j+p_N)/2}}{\prod_{j < k} \xi(z_k/z_j)} v_+(z_1) \otimes \dots \otimes v_+(z_N). \tag{3.2}$$

Here $\xi(z) = (q^2 z; q^4)_\infty / (q^4 z; q^4)_\infty$.

The following proposition is a direct consequence of the properties of the series $F_{\varepsilon_1, \dots, \varepsilon_N}(z)$.

Proposition 3.1. The coefficients of $\Omega(z)$ are well-defined in \hat{V}/\hat{N} . $\Omega(z)$ is symmetric in z_1, \dots, z_N and does not comprise negative powers of them.

The Macdonald polynomials $P_\lambda(z; p, t)$ with $l(\lambda) \leq N$ constitute a basis of symmetric polynomials in z_1, \dots, z_N (see appendix B). Hence any symmetric formal power series has a unique expansion in terms of them. Define vectors $\tilde{\omega}_{\lambda,N} \in \hat{V}/\hat{N}$ by

$$\Omega(z) = \sum_{l(\lambda) \leq N} P_\lambda(z; q^4, q^2) \tilde{\omega}_{\lambda,N}. \tag{3.3}$$

We will determine the Drinfeld polynomial for the $U'_{k=0}$ -module generated by $\tilde{\omega}_{\lambda,N}$. Below we drop the index of $\bar{L}_0(x)$ when there is no chance of confusion.

Proposition 3.2. Let $\bar{L}(x)$ be as in 2.17. Then

$$\bar{L}(x)\Omega(z) = \begin{pmatrix} \hat{A}(x) \cdot \Omega(z) & * \\ 0 & 1 \cdot \Omega(z) \end{pmatrix} \tag{3.4}$$

$$\hat{A}(x) = q^N \frac{\hat{\Delta}(q^{-2}x^{-1})}{\hat{\Delta}(x^{-1})} \quad \hat{\Delta}(u) = \prod_{j=1}^N (1 - q^N \hat{Y}_j^{-1} u). \tag{3.5}$$

The action of $\hat{\Delta}(u)$ on $\tilde{\omega}_{\lambda,N}$ is given as follows.

Proposition 3.3. We have

$$\hat{\Delta}(u)\tilde{\omega}_{\lambda,N} = \Delta_{\lambda,N}(u)\tilde{\omega}_{\lambda,N} \tag{3.6}$$

$$\Delta_{\lambda,N}(u) = \prod_{j=1}^N (1 - q^{-4d_k+1} u) \quad d_k = \lambda_k + \frac{1}{2}(N - k). \tag{3.7}$$

Proof. Since the action of \hat{Y}_j is defined as Y_j on the series $F_{\varepsilon_1, \dots, \varepsilon_N}(z)$, we have

$$\begin{aligned} \hat{\Delta}(u)\Omega(z) &= \prod_{j=1}^N (1 - q^N \hat{Y}_j^{-1} u) \Omega(z) \\ &= \gamma(z)^{-1} \prod_{j=1}^N (1 - q^N \hat{Y}_j^{-1} u) F_{+\dots+}(z) \\ &= \left(\gamma(z)^{-1} \prod_{j=1}^N (1 - q^N Y_j^{-1} u) \gamma(z) \right) \Omega(z). \end{aligned}$$

Under the conjugation by $\gamma(z)$, the operators $G_{j,j+1}$ and Y_j change to

$$g_{j,j+1}^{\pm 1} = \gamma(z)^{-1} \circ G_{j,j+1}^{\pm 1} \circ \gamma(z) = \frac{qz_j - q^{-1}z_{j+1}}{z_j - z_{j+1}}(1 - K_{j,j+1}) - q^{\mp 1}$$

$$y_j = (-q^2)^{-N+1} \gamma(z)^{-1} \circ Y_j \circ \gamma(z) = g_{j,j+1}^{-1} \cdots g_{N-1,N}^{-1} Z g_{1,2} \cdots g_{j-1,j}. \tag{3.8}$$

Here we have used $p = q^4$. Substituting these into (3.3) we obtain

$$\sum_{l(\lambda) \leq N} P_\lambda(z; q^4, q^2) \hat{\Delta}(u) \tilde{\omega}_{\lambda,N} = \sum_{l(\lambda) \leq N} \left(\prod_{j=1}^N (1 + (-1)^N q^{2-N} y_j^{-1} u) P_\lambda(z; q^4, q^2) \right) \tilde{\omega}_{\lambda,N}.$$

On the other hand, the formulae (B.3) and (B.6) yield

$$\prod_{j=1}^N (1 + y_j u) P_\lambda(z; p, q^2) = \prod_{j=1}^N (1 - (-1)^N q^{N-2j+1} p^{j/2} u) P_\lambda(z; p, q^2).$$

The eigenvalues of an arbitrary symmetric function in y_j are determined from this formula. \square

From (3.4)–(3.6) we see that $\tilde{\omega}_{\lambda,N}$ is a highest weight vector whose Drinfeld polynomial is given by (3.7).

3.2. Bosonic expression

Consider now the image of $\tilde{\omega}_{\lambda,N}$ in \mathcal{H} and call it $\omega_{\lambda,N}$:

$$\hat{V}/\hat{N} \rightarrow \hat{V}/\mathcal{N} \simeq \mathcal{H} \quad \tilde{\omega}_{\lambda,N} \mapsto \omega_{\lambda,N}.$$

In the next section we shall study the submodules of \mathcal{H} generated by $\omega_{\lambda,N}$.

As noted in [3] in the case $q = 1$, these vectors can be expressed neatly in terms of bosons. Recall the bosonic realization of the level-1 integrable modules

$$V(\Lambda_i) = \mathbb{C}[b_{-1}, b_{-2}, \dots] \otimes \left(\bigoplus_{n \in \mathbb{Z}} \mathbb{C} \varepsilon^{\Lambda_i + n\alpha} \right).$$

Here b_n 's stand for the standard bosonic oscillators satisfying $[b_m, b_n] = m\delta_{m+n,0}$ ($m, n \in \mathbb{Z} \setminus \{0\}$). The highest weight vector of $V(\Lambda_i)$ is given by $|i\rangle = 1 \otimes \varepsilon^{\Lambda_i}$. In this language the '+'-component of the vertex operator reads as

$$\tilde{\Phi}_+^*(z) = \exp \left(\sum_{n=1}^{\infty} \frac{b_{-n}}{n} q^{2n} z^n \right) \exp \left(- \sum_{n=1}^{\infty} \frac{b_n}{n(1+q^{2n})} z^{-n} \right) \otimes \varepsilon^{\Lambda_i/2} (-qz)^{(\partial+I)/2} (-q)^I.$$

Here I acts as i times the identity on $V(\Lambda_i)$ ($i = 0, 1$). For the details see [13].

Remark. Following [9] we have rescaled the bosons a_n in [13] as

$$a_n = q^{-n/2} \frac{[n]}{n} b_n \quad a_{-n} = q^{n/2} \frac{[2n]}{n} b_{-n} \quad (n > 0).$$

\square

Comparing the generating series with the duality property of the Macdonald polynomials (B.5), we find that

$$\omega_{\lambda,N} = (-q)^M \tilde{P}_{\lambda'}(s; q^2, q^4) \otimes \varepsilon^{\Lambda_i + N\alpha/2} \quad M = \frac{N(N-1)}{2} + \frac{3}{2} \left\lfloor \frac{N}{2} \right\rfloor \tag{3.9}$$

wherein the variables s_n are substituted by the oscillators

$$s_n = (-1)^{n-1} q^{2n} b_{-n}$$

and $[x]$ denotes the integer part of x . From (3.9) it is clear that the vectors $\omega_{\lambda,N} \in \mathcal{H}$ have the homogeneous degree

$$\text{deg } \omega_{\lambda,N} = |\lambda| + \left(\frac{N}{2}\right)^2 - \frac{i}{4} \tag{3.10}$$

where $|\lambda| = \lambda_1 + \dots + \lambda_N$ and $N \equiv i \pmod{2}$ ($i = 0, 1$).

4. Decomposition

In this section, we study how the level-1 integrable modules $V(\Lambda_i)$ ($i = 0, 1$) decompose with respect to the new level-0 action. We assume $|q| < 1$ and that q is transcendental over \mathbb{Q} .

4.1. Submodules

Consider the subspace generated by $\omega_{\lambda,N}$'s,

$$\sum_{\lambda,N} W'_{\lambda,N} \subset \mathcal{H} \quad W'_{\lambda,N} = U'_{k=0} \omega_{\lambda,N} \tag{4.1}$$

where $N \geq 0$ and λ runs over partitions with $l(\lambda) \leq N$.

As the first step we show:

Proposition 4.1. The sum (4.1) is direct.

Lemma 4.2. Let I_1 be the operator acting on $\hat{\mathcal{V}}_N$ by

$$I_1 = \sum_{j=1}^N q^{N-1} \hat{Y}_j^{-1} - \frac{N}{2}(1 + q^2). \tag{4.2}$$

Then it is well-defined on \mathcal{H} and commutes with the action of $U'_{k=0}$.

Proof. By the construction of the action of $U'_{k=0}$, symmetric polynomials in \hat{Y}_j act on $\hat{\mathcal{V}}_N / (\tilde{\mathcal{N}} \cap \hat{\mathcal{V}}_N)$, commuting with $U'_{k=0}$. For (4.2) to be well-defined on \mathcal{H} , we must show that it is compatible with the fusion relation. Let us set

$$I_1^{(NY)} = \sum_{j=1}^N q^{N-1} \left(\hat{Y}_j^{(N)} \right)^{-1} \tag{4.3}$$

where the N -dependence is exhibited explicitly. For the action to be well-defined, the following relations must hold:

$$\begin{aligned} & \left(I_1^{(NY)} F_{\varepsilon_1 \dots \varepsilon_N}(z_1, \dots, z_N) \right) \Big|_{z_N = q^{-2} z_{N-1}} \\ &= \left(I_1^{(N-2)'} + 1 + q^2 \right) \left(F_{\varepsilon_1 \dots \varepsilon_N}(z_1, \dots, z_N) \Big|_{z_N = q^{-2} z_{N-1}} \right). \end{aligned}$$

This can be verified in almost the same way as in the proof of proposition 12 (4.8) in [12]. \square

Proof of Proposition 4.1. Since I_1 commutes with U' , it acts as a scalar on $W'_{\lambda,N}$. From the formula (3.6), its eigenvalue is given by

$$\sum_{j=1}^N q^{-4d_j} - \frac{N}{2}(1 + q^2).$$

Note that $d_1 > \dots > d_N \geq 0$. Since we assume q to be transcendental, these eigenvalues are all distinct when (λ, N) runs over all possible pairs. The proposition follows from this fact. \square

4.2. Irreducible modules

Let $W_{\lambda,N}$ denote the irreducible module which has the same Drinfeld polynomial (3.6) as $\omega_{\lambda,N}$. At this stage we can only say that

$$\dim W'_{\lambda,N} \geq \dim W_{\lambda,N}. \tag{4.4}$$

This holds because the irreducible module has the minimal dimension among the finite dimensional highest weight modules with the same highest weight. As the next step, we study the structure of $W_{\lambda,N}$.

Given λ with $l(\lambda) \leq N$, let us re-parametrize it by two sets of non-negative integers $\{n_i\}$ and $\{k_i\}$ as follows:

$$\lambda = (\underbrace{k_1, \dots, k_1}_{n_1}, \underbrace{k_2, \dots, k_2}_{n_2}, \dots, \underbrace{k_m, \dots, k_m}_{n_m}). \tag{4.5}$$

Here m is a positive integer, $k_1 > k_2 > \dots > k_m \geq 0$, and $\sum_{i=1}^m n_i = N$. Set

$$a_j = q^{-4k_j - 2N + 2(n_1 + \dots + n_{j-1}) + n_j + 1}. \tag{4.6}$$

Then the Drinfeld polynomial $\Delta_{\lambda,N}(u)$ in (3.6) is factorized as

$$\Delta_{\lambda,N}(u) = \Delta_{n_1}(u; a_1) \Delta_{n_2}(u; a_2) \dots \Delta_{n_m}(u; a_m)$$

with

$$\Delta_n(u; a) = \prod_{i=1}^n (1 - q^{n-2(i-1)} au).$$

To each polynomial $\Delta_{n_j}(u; a_j)$, one can associate the $(n_j + 1)$ -dimensional evaluation module $W_{n_j}(a_j)$ of $U'_{k=0}$ (see appendix A for the conventions). The parametrization (4.5)–(4.6) is so designed that for any distinct pair (n_j, a_j) , (n_k, a_k) the tensor product $W_{n_j}(a_j) \otimes W_{n_k}(a_k)$ is irreducible. This implies [6] the following proposition.

Proposition 4.3.

$$W_{\lambda,N} \simeq W_{n_1}(a_1) \otimes \dots \otimes W_{n_m}(a_m). \tag{4.7}$$

4.3. Characters

For $i = 0, 1$, define

$$\mathcal{V}^{(i)} \equiv \bigoplus_{\substack{N \equiv i \pmod{2} \\ l(\lambda) \leq N}} W'_{\lambda,N} \subset \mathcal{H}. \tag{4.8}$$

The third step is to estimate its character

$$\text{ch}_{\mathcal{V}^{(i)}}(q) = \text{tr}_{\mathcal{V}^{(i)}}(q^{-d}) = \sum_{n \geq 0} (\dim \mathcal{V}_{-n}^{(i)}) q^n \tag{4.9}$$

where d denotes the homogeneous grading operator on \mathcal{H} . Given formal series $f(q) = \sum f_n q^n$ and $g(q) = \sum g_n q^n$, we write $f(q) \gg g(q)$ to mean $f_n \geq g_n$ for all n .

Lemma 4.4.

$$\text{ch}_{\chi^{(0)}}(q) \gg \sum_{\substack{N \equiv i \pmod{2} \\ l(\lambda) \leq N}} q^{\frac{N^2}{4} - \frac{i}{4} + |\lambda|} \prod_{j=1}^m (n_j + 1). \tag{4.10}$$

Proof. The degree of $\omega_{\lambda,N}$ is given by (3.10). Since the action of $U'_{k=0}$ commutes with d , this value of the degree occurs with multiplicity $\dim W'_{\lambda,N}$. From (4.4) and (4.7) we obtain

$$\dim W'_{\lambda,N} \geq \dim W_{\lambda,N} = \prod_{i=1}^m (n_i + 1).$$

The estimate (4.10) follows from this. □

It is known [1, 3, 17] that the right-hand side of (4.10) gives the (homogeneous) character for the level-1 module $V(\Lambda_i)$. This leads us to the main result.

Theorem 4.5. We have $W'_{\lambda,N} \simeq W_{\lambda,N}$. The level-1 integrable highest weight module $V(\Lambda_i)$ is completely reducible under the action of $U'_{k=0}$:

$$V(\Lambda_i) \simeq \bigoplus_{\substack{N \equiv i \pmod{2} \\ l(\lambda) \leq N}} W_{\lambda,N} \quad (i = 0, 1). \tag{4.11}$$

Remark. The action of the $U_q(sl_2)$ is common to both level-0 and level-1 structures. Denoting by \hbar the standard generator of the Cartan subalgebra of sl_2 , we have therefore

$$\text{tr}_{W'_{\lambda,N}}(q^d z^{\hbar}) = q^{\frac{N^2}{4} - \frac{i}{4} + |\lambda|} \prod_{j=1}^m \chi_{n_j}(z)$$

where

$$\chi_n(z) = \frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}} = \sum_{\substack{n^+, n^- \in \mathbb{Z}_{\geq 0} \\ n^+ + n^- = n}} z^{n^+ - n^-}$$

is the character of the $(n+1)$ -dimensional irreducible representation of sl_2 . The full character is then given by

$$\text{ch}_{\chi^{(0)}}(q, z) = \sum_{\substack{n^+, n^- \in \mathbb{Z}_{\geq 0} \\ n^+ + n^- \equiv i \pmod{2}}} z^{n^+ - n^-} \frac{q^{\frac{(n^+ + n^-)^2}{4} - \frac{i}{4}}}{(q; q)_{n^+} (q; q)_{n^-}}$$

with $(q; q)_n = (1 - q) \cdots (1 - q^n)$. This agrees with equation (5.2) in [16]. □

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Appendix A. Drinfeld polynomials

The aim of this appendix is to summarize our conventions concerning various generators of $U' = U'(\widehat{sl}_2)$ and Drinfeld polynomials.

A.1. Algebra \tilde{U}'

Following [8, 18], define $\tilde{U}' = U'_q(\widehat{gl}_2)$ to be the associative algebra with the unit generated by the symbols $l_{ij}^\pm[\pm n]$ ($i, j = 1, 2, n \geq 0$) and an invertible central element $q^{c/2}$. In terms of the matrix generating series

$$L^\pm(x) = \sum_{\pm n \geq 0} x^n \begin{pmatrix} l_{11}^\pm[n] & l_{12}^\pm[n] \\ l_{21}^\pm[n] & l_{22}^\pm[n] \end{pmatrix} \tag{A.1}$$

the defining relations of \tilde{U}' read as follows

$$\begin{aligned} l_{ii}^+[0]l_{ii}^-[0] &= 1 \quad (i = 1, 2) & l_{21}^+[0] &= l_{12}^-[0] = 0 \\ \overline{R}_{00'}(x_0/x_0')L_0^\pm(x_0)L_0^\pm(x_0') &= L_0^\pm(x_0')L_0^\pm(x_0)\overline{R}_{00'}(x_0/x_0') \\ \overline{R}_{00'}(q^c x_0/x_0')L_0^+(x_0)L_0^-(x_0') &= L_0^-(x_0')L_0^+(x_0)\overline{R}_{00'}(q^{-c} x_0/x_0') \end{aligned}$$

where

$$\overline{R}(x) = \begin{pmatrix} 1 & & & \\ & \frac{(1-z)q}{1-q^2z} & \frac{(1-q^2)}{1-q^2z} & \\ & \frac{(1-q^2)z}{1-q^2z} & \frac{(1-z)q}{1-q^2z} & \\ & & & 1 \end{pmatrix}. \tag{A.2}$$

The matrices (A.1) can be uniquely decomposed [8] as

$$\begin{aligned} L^\pm(x) &= \begin{pmatrix} 1 & f^\pm(x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k_1^\pm(x) & 0 \\ 0 & k_2^\pm(x) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^\pm(x) & 1 \end{pmatrix} \\ k_i^\pm(x) &= \sum_{\pm n \geq 0} k_{i,n}^\pm x^n & e^\pm(x) &= \sum_{\pm n \geq 0} e_n^\pm x^n & f^\pm(x) &= \sum_{\pm n \geq 0} f_n^\pm x^n \end{aligned}$$

with $e_0^+ = f_0^- = 0$. Define $\tilde{x}_k^\pm, \tilde{\psi}_k, \tilde{\varphi}_{-k}$ by

$$\begin{aligned} k_1^+(x)k_2^+(x)^{-1} &= \sum_{k=0}^\infty \tilde{\varphi}_{-k} x^k & k_1^-(x)k_2^-(x)^{-1} &= \sum_{k=0}^\infty \tilde{\psi}_k x^{-k} \\ \frac{-1}{q-q^{-1}} (e^+(q^{c/2}x) - e^-(q^{-c/2}x)) &= \sum_{n \in \mathbb{Z}} \tilde{x}_n^+ x^{-n} \\ \frac{-1}{q-q^{-1}} (f^+(q^{-c/2}x) - f^-(q^{c/2}x)) &= \sum_{n \in \mathbb{Z}} \tilde{x}_n^- x^{-n}. \end{aligned}$$

Let x_k^\pm ($k \in \mathbb{Z}$), ψ_k, φ_{-k} ($k \in \mathbb{Z}_{\geq 0}$), $K = q^h$ and γ be the Drinfeld generators of U' in the notation of [13].

Proposition A.1 ([8]). U' is isomorphic to a subalgebra of \tilde{U}' by the map

$$\psi_k \mapsto \tilde{\psi}_k \quad \varphi_k \mapsto \tilde{\varphi}_k \quad x_n^\pm \mapsto \tilde{x}_n^\pm \quad \gamma \mapsto q^c.$$

Define $l_{ij}^\pm(x) = \sum_{\pm n \geq 0} x^n l_{ij}^\pm[n]$. The coproduct on \tilde{U}' is defined by

$$\Delta' \left(l_{ij}^\pm(x) \right) = \sum_{k=1,2} l_{ik}^\pm(q^{\pm c_2/2}x) \otimes l_{kj}^\pm(q^{\mp c_1/2}x)$$

where $c_1 = c \otimes 1$ and $c_2 = 1 \otimes c$. In terms of the Chevalley generators

$$\begin{aligned} e_0 &= x_1^- t_0 & f_0 &= t_0^{-1} x_{-1}^+ & t_0 &= \gamma K^{-1} \\ e_1 &= x_0^+ & f_1 &= x_0^- & t_1 &= K \end{aligned} \tag{A.3}$$

the coproduct induced on U' reads

$$\Delta'(e_i) = e_i \otimes t_i + 1 \otimes e_i \quad \Delta'(f_i) = f_i \otimes 1 + t_i^{-1} \otimes f_i \quad \Delta'(t_i) = t_i \otimes t_i.$$

The identification of e_0, f_0 in (A.3) differs from the one in [13] by a factor of γ . Note also that $\Delta' = \Delta^{op}$.

A.2. Drinfeld polynomials

Let us recall from [6] some basic facts about the finite dimensional representations of U' . On a finite-dimensional U' -module γ acts as 1.

Let W be a finite dimensional U' -module. For a polynomial $P(u) \in \mathbb{C}[u]$ satisfying $P(0) = 1$, let $d_k^\pm \in \mathbb{C}$ be the coefficients in the expansion

$$\begin{aligned} q^{\deg P} \frac{P(q^{-2}u)}{P(u)} &= \sum_{k \geq 0} d_k^+ u^k && \text{(as } u \rightarrow 0) \\ &= \sum_{k \leq 0} d_k^- u^k && \text{(as } u \rightarrow \infty). \end{aligned}$$

A vector $w \in W$ is said to be a highest weight vector with the Drinfeld polynomial $P(u)$ if the following hold:

$$x_k^+ w = 0 \quad (k \in \mathbb{Z}) \tag{A.4}$$

$$\psi_k w = d_k^+ w \quad (k \geq 0) \quad \varphi_k w = d_k^- w \quad (k \leq 0). \tag{A.5}$$

Associating with w its Drinfeld polynomial, one obtains a bijective correspondence between the equivalence class of irreducible finite dimensional modules and the set of polynomials normalized as $P(0) = 1$.

Suppose the module W is defined through (A.1). If $w \in W$ has the property

$$L^\pm(x)w = \begin{pmatrix} A^\pm(x)w & * \\ 0 & D^\pm(x)w \end{pmatrix}$$

with some functions $A^\pm(x), D^\pm(x)$, then the conditions (A.4), (A.5) are satisfied. The Drinfeld polynomial $P(u)$ is determined by

$$\begin{aligned} q^{\deg P} \frac{P(q^{-2}u)}{P(u)} &= \frac{A^+(u^{-1})}{D^+(u^{-1})} && \text{(as } u \rightarrow 0) \\ &= \frac{A^-(u^{-1})}{D^-(u^{-1})} && \text{(as } u \rightarrow \infty). \end{aligned}$$

Example. Let W_n be the $(n + 1)$ -dimensional irreducible module of $U_q(sl_2)$. By $W_n(a)$ we mean the evaluation module with the parameter $a \in \mathbb{C} \setminus \{0\}$ whose U' -module structure is given by

$$e_0 = af_1 \quad f_0 = a^{-1}e_1 \quad t_0 = t_1^{-1}.$$

In our convention, the corresponding Drinfeld polynomial is

$$P_n(u; a) = (1 - q^n au)(1 - q^{n-2} au) \dots (1 - q^{-n+2} au).$$

Appendix B. Macdonald polynomials

B.1. Macdonald polynomials

We recall here some facts concerning the Macdonald polynomials which are used in the text. For details see [14, 15]. By a partition we mean a finite sequence of non-increasing integers $\lambda = (\lambda_1, \dots, \lambda_N)$, $\lambda_1 \geq \dots \geq \lambda_N \geq 0$. Its length is $l(\lambda) = \max\{j \mid \lambda_j > 0\}$.

Let Λ_N be the ring of symmetric polynomials in N variables z_1, z_2, \dots, z_N with coefficients in $\mathbb{Q}(p, t)$, where p, t are indeterminates. (In the usual notation the letter q is used in place of p .) As before write z for (z_1, z_2, \dots, z_N) . The Macdonald polynomials $P_\lambda(z; p, t)$ are a certain basis of Λ_N indexed by partitions λ with $l(\lambda) \leq N$.

The Macdonald polynomials are eigenfunctions of the Macdonald operators defined as follows. For each $n = 1, \dots, N$, let

$$D_N^n(p, t) = t^{\frac{n(n-1)}{2}} \sum_I \left(\prod_{\substack{j \in I \\ j \notin I}} \frac{tz_j - z_j}{z_j - z_j} \right) \prod_{k \in I} p^{\theta_k}. \tag{B.1}$$

Here the summation is taken over all n -element subsets I of $\{1, 2, \dots, N\}$, and p^{θ_j} denotes the shift operator

$$(p^{\theta_j} f)(z_1, \dots, z_N) = f(z_1, \dots, pz_j, \dots, z_N).$$

Setting $D_N^0(p, t) = 1$, we consider the generating function

$$D(u; p, t) = \sum_{n=0}^N D_N^n(p, t) u^n. \tag{B.2}$$

Proposition B.1.

$$D(u; p, t) P_\lambda(z; p, t) = \prod_{j=1}^N (1 + t^{N-j} p^{\lambda_j} u) P_\lambda(z; p, t). \tag{B.3}$$

In particular

$$D_N^1(p, t) P_\lambda(z; p, t) = \left(\sum_{j=1}^N t^{N-j} p^{\lambda_j} \right) P_\lambda(z; p, t). \tag{B.4}$$

Let $\Lambda = \varinjlim \Lambda_N$ be the projective limit given via the restriction $\Lambda_{N+1} \rightarrow \Lambda_N$, $P \mapsto P|_{z_{N+1}=0}$. For each fixed λ , $P_\lambda(z; p, t)$ gives rise to a well-defined element in Λ . We denote it by $\tilde{P}_\lambda(s; p, t) \in \Lambda$ viewed as a polynomial in the power sum $s_n = \sum_j z_j^n$, $n \geq 1$. Rewriting the duality property ([14], equation (3.12)) we have

Proposition B.2. Let λ' denote the partition conjugate to λ (given by transposing the corresponding Young diagram). Then

$$\exp \left(\sum_{n \geq 1} (-1)^{n-1} \frac{s_n}{n} \sum_{j=1}^N w_j^n \right) = \sum_{l(\lambda) \leq N} P_\lambda(w; p, t) \tilde{P}_{\lambda'}(s; t, p) \tag{B.5}$$

where the sum ranges over all partitions of length at most N .

B.2. A lemma on Macdonald operators

Consider the operators $y_j, r_{ij} = K_{ij}g_{ij}$ given in (3.8):

$$\begin{aligned}
 y_j &= r_-(j, N)p^{\theta_j}r_+(1, j) \\
 r_-(i, j) &= r_{ii+1}^{-1}r_{ii+2}^{-1} \cdots r_{ij}^{-1} \\
 r_+(i, j) &= r_{ij}r_{i+1j} \cdots r_{j-1j} \\
 r_{ij} &= b_{ji}(1 - K_{ij}) - q^{-1}K_{ij} \quad r_{ij}^{-1} = b_{ij}(1 - K_{ij}) - qK_{ij} \\
 b_{ij} &= -\frac{qz_i - q^{-1}z_j}{z_i - z_j} \quad \dots \quad b_{ij} + b_{ji} = -q - q^{-1}.
 \end{aligned}$$

Here we regard p, q as independent (we do not restrict to $p = q^4$). In this subsection we prove the following proposition.

Proposition B.3. For any symmetric polynomial $P \in \Lambda_N$, we have

$$\prod_{j=1}^N (1 + y_j u) P = D((-q)^{-N+1} u; p, q^2) P \tag{B.6}$$

where $D(u; p, t)$ is given in (B.2).

Let \mathcal{L} be a homogeneous polynomial in y_j of degree n . When applied to a symmetric polynomial $P \in \Lambda_N$, we can move all the K_{ij} to the right and rewrite $\mathcal{L}P$ in the form

$$\tilde{\mathcal{L}}P = \sum_{1 \leq j_1 < \dots < j_n \leq N} f_{j_1, \dots, j_n}(z) p^{\theta_{j_1}} \cdots p^{\theta_{j_n}} P \tag{B.7}$$

with some rational functions $f_{j_1, \dots, j_n}(z)$. We shall refer to $\tilde{\mathcal{L}}$ as the normal form of \mathcal{L} . The following shows that it is uniquely determined from \mathcal{L} .

Lemma B.4. Suppose an operator $\tilde{\mathcal{L}}$ of the form (B.7) satisfies $\tilde{\mathcal{L}}P = 0$ for all $P \in \Lambda_N$, identically in p . Then $f_{j_1, \dots, j_n}(z) = 0$ for all j_1, \dots, j_n .

Proof. It is helpful to extend the definition of $f_{j_1, \dots, j_n}(z)$ for all $j_1, \dots, j_n \in \{1, \dots, N\}$ by symmetry in j_1, \dots, j_n , and as 0 whenever two indices coincide.

Take $P = \sum_{\sigma \in S_N} z_{\sigma(1)}^{\lambda_1} \cdots z_{\sigma(n)}^{\lambda_n}$ where λ_j 's are arbitrary positive integers. By assumption $\tilde{\mathcal{L}}P = 0$. Picking the coefficient of $p^{\lambda_1 + \dots + \lambda_n}$ we find

$$\sum_{j_1, \dots, j_n=1}^N f_{j_1, \dots, j_n}(z) z_{j_1}^{\lambda_1} \cdots z_{j_n}^{\lambda_n} = 0. \tag{B.8}$$

Now if $\sum_{i=1}^N z_i^{\lambda} g_i(z) = 0$ holds for any $\lambda = 1, 2, \dots$, then $g_i(z) = 0$ for all i . Since λ_j 's are arbitrary, (B.8) implies $\sum_{j_2, \dots, j_n} f_{j_1, \dots, j_n}(z) z_{j_2}^{\lambda_2} \cdots z_{j_n}^{\lambda_n} = 0$ for all j_1 . Repeating this process we obtain the lemma. □

From the lemma, we must have $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}^\sigma$ for any $\sigma \in S_N$, where

$$\tilde{\mathcal{L}}^\sigma = \sum_{1 \leq j_1 < \dots < j_n \leq N} f_{j_1, \dots, j_n}(z_{\sigma(1)}, \dots, z_{\sigma(N)}) p^{\theta_{\sigma(j_1)} + \dots + \theta_{\sigma(j_n)}}.$$

Therefore $\tilde{\mathcal{L}}$ is determined once $f_{1, 2, \dots, n}(z)$ is known.

For each $n = 1, \dots, N$ define

$$\mathcal{L}_n = \sum_{1 \leq j_1 < \dots < j_n \leq N} y_{j_1} \cdots y_{j_n}.$$

Then proposition B.3 is equivalent to the following.

Proposition B.5. Let $I = \{1, 2, \dots, N\}$. The normal form of \mathcal{L}_n is given by

$$\tilde{\mathcal{L}}_n = \sum_{|I|=n} \prod_{\substack{i \in I \\ j \notin I}} b_{ij} \prod_{i \in I} p^{\vartheta_i} = (-q)^{-n(N-1)} D_N^n(p, q^2) \quad (\text{B.9})$$

where $D_N^n(p, t)$ signifies the Macdonald operator (B.1).

Proof. We prove B.9 by induction on n . The case $n = 0$ is trivially true.

From the remark above it suffices to compute the coefficient of $p^{\vartheta_1 + \dots + \vartheta_n}$ in $\tilde{\mathcal{L}}_n$. If $j_1 < \dots < j_n$, then we have

$$y_{j_1} \cdots y_{j_n} = r_-(j_1, N) p^{\vartheta_{j_1}} r_-(j_2, N) p^{\vartheta_{j_2}} \cdots r_-(j_n, N) p^{\vartheta_{j_n}} r_+(1, j_1) r_+(2, j_2) \cdots r_+(n, j_n). \quad (\text{B.10})$$

Suppose further $2 \leq j_1 < \dots < j_n$, and set $y'_j = r_-(j, N) p^{\vartheta_j} r_+(2, j)$. We then find, using $r_{ij} P = -q^{-1} P$ for $P \in \Lambda_N$, that

$$y_{j_1} \cdots y_{j_n} P = (-q)^{-n} y'_{j_1} \cdots y'_{j_n} P. \quad (P \in \Lambda_N). \quad (\text{B.11})$$

Let us write $\mathcal{L}_n = y_1 \mathcal{L}'_{n-1} + \mathcal{L}''_n$, where

$$\mathcal{L}'_{n-1} = \sum_{2 \leq j_2 < \dots < j_n \leq N} y_{j_2} \cdots y_{j_n}, \quad \mathcal{L}''_n = \sum_{2 \leq j_1 < \dots < j_n \leq N} y_{j_1} \cdots y_{j_n}.$$

From (B.10) it is clear that the normal forms of \mathcal{L}'_{n-1} , \mathcal{L}''_n do not contain terms $\prod_{i \in I} p^{\vartheta_i}$ with $1 \in I$. To calculate the coefficient of $p^{\vartheta_1 + \dots + \vartheta_n}$, it is therefore enough to consider $y_1 \mathcal{L}'_{n-1}$. By (B.11) we may apply the induction hypothesis to \mathcal{L}'_{n-1} , to obtain

$$\begin{aligned} y_1 \mathcal{L}'_{n-1} P &= r_-(1, n) \mathcal{L}''' P \quad (P \in \Lambda_N) \\ \mathcal{L}''' &= (-q)^{-n+1} r_{1, n+1}^{-1} \cdots r_{1, N}^{-1} p^{\vartheta_1} \sum_{I'} \prod_{\substack{i \in I' \\ j \notin I'}} b_{ij} \prod_{i \in I'} p^{\vartheta_i}. \end{aligned}$$

Here the sum $\sum_{I'}$ ranges over the subsets $I' \subset \{2, \dots, N\}$ with $n-1$ elements. Let us bring the K_{ij} contained in r_{ij} to the right and pick the coefficient of $p^{\vartheta_1 + \dots + \vartheta_n}$ in the normal form of $r_-(1, n) \mathcal{L}'''$. The operator $r_-(1, n)$ comprises permutations of z_1, \dots, z_n alone, so the coefficient of $p^{\vartheta_1 + \dots + \vartheta_n}$ comes from that of \mathcal{L}''' . It is easy to see that its normal form is

$$\tilde{\mathcal{L}}''' = (-q)^{-n+1} \prod_{j=n+1}^N b_{1j} \prod_{\substack{2 \leq i \leq n \\ n < j \leq N}} b_{ij} \times \prod_{i=1}^n p^{\vartheta_i} + \dots$$

with \dots denoting terms containing p^{ϑ_i} with $i > n$. The first term is symmetric in the indices $\{1, \dots, n\}$, so r_{1j}^{-1} ($2 \leq j \leq n$) acts as $-q$. Hence the coefficient of $p^{\vartheta_1 + \dots + \vartheta_n}$ in $\tilde{\mathcal{L}}$ is given by

$$(-q)^{n-1} (-q)^{-n+1} \prod_{\substack{1 \leq i \leq n \\ n < j \leq N}} b_{ij} = \prod_{\substack{1 \leq i \leq n \\ n < j \leq N}} b_{ij}.$$

This completes the proof. \square

Note added in proof. There is a small mistake in [12] relevant to this paper. In subsection 4.2 of [12], the last line of the second to last equation should read:

$$= (S_{j,j+1} - \hat{G}_{j,j+1})q^{N-1}(\hat{Y}_j^{-1}f^{(j+1)} + \hat{Y}_j^{-1}f^{(j+1)}).$$

The following text should then be inserted:

Hence

$$\begin{aligned} \pi^{(N)}(\epsilon_0)(S_{j,j+1} - \hat{G}_{j,j+1})F_{\epsilon_1 \dots \epsilon_N}(z_1, \dots, z_N) &= (S_{j,j+1} - \hat{G}_{j,j+1}) \\ &\times \left(\sum_{k(\neq j,j+1)} e_0^{(k)} + \hat{Y}_j^{-1}f^{(j+1)} + \hat{Y}_j^{-1}f^{(j+1)} \right) F_{\epsilon_1 \dots \epsilon_N}(z_1, \dots, z_N) \sim 0 \end{aligned}$$

which completes the proof for ϵ_0 .

The last equation in subsection 4.2 of [12] should read $S(\epsilon_1 \otimes \epsilon_1) = (\epsilon_1 \otimes 1)S$.

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